Topology in physics 2019, exercises for lecture 14

- The hand-in exercise is exercise 1. Recall that if you did not so so yet, you should also hand in exercise 4 from the exercise set for lecture 12 this week.
- If you have not made exercise 1 of lecture 9 yet (this was not a hand-in exercise), we suggest taking a look at that as well as it is very relevant for the material in this lecture.
- Please hand in electronically at `topologyinphysics2019@gmail.com` (1 pdf, readable!)
- Deadline is Wednesday May 22, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercises

⋆ Exercise 1: Zeta-function regularization

Let $\mathcal{O}$ be an operator acting on a Hilbert space, with a complete set of eigenstates v_n with eigenvalues λ_n ($n = 1, 2, 3, \ldots$). We introduce its *spectral ζ-function* as

$$\zeta_{\mathcal{O}}(s) = \sum_{n=1}^{\infty} (\lambda_n)^{-s}. \tag{1}$$

Beware that this means that we are counting with multiplicities so that eg. $\zeta_{\lambda I_n}(s) = \frac{n}{\lambda^s}$. Note that in the special case where $\lambda_n = n$, this function is the ordinary Riemann zeta function $\zeta(s)$. As is the case for that function, we will assume in what follows that $\zeta_{\mathcal{O}}(s)$ is well-defined for $\mathrm{Re}(s)$ large enough, and that it can then be analytically continued to a meromorphic function on the complex s-plane.

a. Show that

$$\det \mathcal{O} = e^{-\zeta'_{\mathcal{O}}(0)} \tag{2}$$

whenever both sides of this equation are well-defined.

Zeta-function regularization now *defines* $\det \mathcal{O}$ by the right hand side of the above equation (using the analytic contnuation of $\zeta_{\mathcal{O}}(s)$) whenever it is not well-defined directly as a product of the eigenvalues of $\mathcal{O}$.

We are now interested in the situation where

$$\mathcal{O} = -\frac{d^2}{dt^2} \tag{3}$$

where $t \in [0, \beta]$ parameterizes a circle of circumference β. (Note that we are using periodic boundary conditions on the eigenstates of $\mathcal{O}$.) To obtain a nonzero and well-defined result, we remove the "zero mode" (the constant eigenfunction of $\mathcal{O}$) from the Hilbert space.

b. Show that, after the above removal,

$$\zeta_{\mathcal{O}}(s) = 2\left(\frac{\beta}{2\pi}\right)^{2s} \zeta(2s) \tag{4}$$

where the function appearing on the right hand side is the ordinary Riemann ζ-function.

c. Show that

$$\det{}'\mathcal{O} = \beta^2 \tag{5}$$

where the prime on the left hand side indicates the removal of the zero mode. You can use the known values of the Riemann ζ-function and its derivative at the origin: $\zeta(0) = -1/2$ and $\zeta'(0) = -\log(2\pi)/2$.

Exercise 2: Product formula for the sine

Since it is so essential in the proof of the index theorem, we want to prove the product formula for the sine,

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right) \tag{6}$$

a. Show that the Fourier series for the function $\cos(\alpha x)$ equals

$$\cos(\alpha x) = \frac{\alpha \sin(\pi\alpha)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\alpha^2 - n^2} \cos(nx). \tag{7}$$

b. Deduce from the above result that

$$\cot(\pi\alpha) - \frac{1}{\pi\alpha} = \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha^2 - n^2} \tag{8}$$

c. Integrate the above formula from $\alpha = 0$ to $\alpha = t$ (you may assume without proof that the sum and integral can be exchanged) and use the result to obtain the product formula for the sine.